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# On the Rainbow and Strong Rainbow Coloring of Comb Product Graphs

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**Abstract:** Let $G=(V,E)$ be a simple, nontrivial, finite, connected and undirected graph. Let $c$ be a coloring $c: E(G) \rightarrow \{1,2,\ldots,k\}$, $k \in N$. A path of the edges of a colored graph is said to be a rainbow path if no two edges on the path have the same color but the adjacent edges may be colored by the same colors. An edge colored graph $G$ is rainbow connected if there exists a rainbow $u-v$ path for every two vertices $u$ and $v$ of $G$. Furthermore, for any two vertices $u$ and $v$ of $G$, a rainbow $u-v$ geodesic in $G$ is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between $u$ and $v$. The graph $G$ is strongly rainbow connected if there exists a rainbow $u-v$ geodesic for any two vertices $u$ and $v$ in $G$. The rainbow and strong rainbow connection numbers of a graph $G$, denoted by $rc(G)$ and $src(G)$ respectively, are the minimum number of colors that are needed in order to make $G$ rainbow and strongly rainbow connected, respectively. Some results have shown the lower and upper bound of $rc(G)$ and $src(G)$, but most of them are not sharp. Thus, finding an exact value of $rc(G)$ and $src(G)$ are significantly useful. In this paper, we study the exact values of rainbow and strong rainbow connection numbers of comb product graphs.

**Keywords:** *rainbow connection; strong rainbow connection; comb product of graphs.*

## 1. Introduction

In this paper, we study a graph which is nontrivial, finite, simple, undirected and connected. For all terminology related to the graph elements, we lead reader to see [8]. Recently, a security of delivery system becomes a main issue. Here we consider a graph representation as a network topology. There are many ways to strengthen the security of delivery system concept, such as requiring hamiltonicity, $k$-connectivity, imposing bounds on the diameter, and so on. Motivated by an edge coloring and security of delivery system design, Chartrand et al. [4] in 2008 introduced an interesting way to strengthen the connectivity requirement, namely a rainbow connection of graph.

Let $c$ be a coloring $c: E(G) \rightarrow \{1,2,\ldots,k\}$, $k \in N$. A path of the edges of a colored graph is said to be a rainbow path if no two edges on the path have the same color but the adjacent edges may be colored by the same colors. An edge colored graph $G$ is rainbow connected if there exists a rainbow $u-v$ path for every two vertices $u$ and $v$ of $G$. Furthermore, for any two vertices $u$ and $v$ of $G$, a rainbow $u-v$ geodesic in $G$ is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between $u$ and $v$. The graph $G$ is strongly rainbow connected if there exists a rainbow $u-v$ geodesic for any two vertices $u$ and $v$ in $G$. The rainbow and strong rainbow connection numbers of a graph $G$, denoted by $rc(G)$ and $src(G)$ respectively, are the minimum number of colors that are needed in order to make $G$ rainbow and strongly rainbow

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connected, respectively. Some results have shown the lower and upper bound of $rc(G)$ and $src(G)$, but most of them are not sharp. Determination of the rainbow or strong rainbow connection numbers for any graph are considered to be a complex problem. Suppose an edge coloring of the graph is given. How do we decide whether the colored graphs are rainbow or strong rainbow connected or not? This problem can not be solved in the polynomial times, it is an NP-Complete problem.

There are some results regarded to rainbow coloring described in [1, 2, 5, 7], but the problems are still open since the study has been much focused on special graphs. In [1], Caro, Lev, Roditty, Tuza and Yuster investigated the upper bound of the rainbow connection number of graphs and they proved that if $G$ is a connected graph of order $n$ and with minimum degree $\delta(G)\geq 3$, then $rc(G)<5n/6$. Schiermeyer [12] sharped the upper bound by proving that if $G$ is a connected graph of order $n$ with $\delta(G)\geq 3$, then $rc(G)<(3n-1)/4$. Chandran et. al. [2] also proved that for every connected graph $G$ of order $n$ and minimum degree $\delta$, $rc(G)\leq(3n+1)/\delta+3$. For complete results of upper bounds of rainbow connection number, the readers are referred to [10]. Dafik et.al. [6] studied the rainbow coloring for some graph operations. Agustin, et.al. [9] determined the rainbow $k$-connection number of special graphs and its sharp lower bound.

There are many results regarded to rainbow connection number, but there are less study focusing into strong rainbow connection number. Thus, finding an exact value of $rc(G)$ and $src(G)$ are useful. In this paper, we study the exact values of rainbow and strong rainbow connection numbers of comb product graphs.

Chandran et al. [2] proved the following useful proposition.

**Proposition 1.** [2]
(i) *For any connected graph $G$ of size $m$, $diam(G)\leq rc(G)\leq src(G)\leq m$, where $diam(G)$ is the diameter of $G$,*
(ii) $rc(G)=1$ *if and only if $G$ is complete,*
(iii) $rc(G)=m$ *if and only if $G$ is a tree of order $m+1$,*
(iv) *For cycle $C_n$, $n\geq 4$, $rc(C_n)=src(C_n)=\lceil n/2\rceil$,*
(v) *For wheel $W_n$ of order $n+1$, $rc(W_n)=1$ if $n=3$; $rc(W_n)=2$ if $4\leq n\leq 6$; $rc(W_n)=3$ if $n\geq 7$.*

Let $G$ and $H$ be two connected graphs. Let $o$ be a vertex of $H$. The comb product between graphs $G$ and $H$, denoted by $G\rhd H$, is a graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and grafting the $i$-th copy of $H$ at the vertex o to the $i$-th vertex of $G$. $|V(G\rhd H)|=|V(G)|.|V(H)|$ and $|E(G\rhd H)|=|V(G)|.|E(H)|+|E(G)|$.

In this paper, we study the exact values of rainbow and strong rainbow connection numbers of comb product of special graphs, namely path and cycle $P_n\rhd C_m$, path and fan graph $P_n\rhd F_m$, and path and triangular book $P_n\rhd Bt_m$.

## 2. Rainbow Coloring

First, we will show the results on the rainbow coloring of graphs.

**Lemma 1.** *Let $P_n$ be a path of order $n$, and $H$ be any connected graph. For comb product of $P_n$ and $H$ holds*

$$diam(P_n\rhd H)\leq rc(P_n\rhd H)\leq n\,.\,rc(H)+n\text{-}1.$$

**Proof.** Let $H$ be any graph of order $p$ and size $q$ and $P_n$ be a path of order $n$. From the definition of the comb product of $P_n$ and $H$ we have that $|V(P_n\rhd H)|=pn$ and $|E(P_n\rhd H)|=qn+n-1$. From Proposition 1 it follows

$$rc(P_n\rhd H)\geq diam(P_n\rhd H). \quad (1)$$

It follows that the rainbow connection number of the comb product $P_n\rhd H$ can not be larger than $n.\ rc(H)+n\text{-}1$. Thus

$$rc(P_n\rhd H)\leq n\,.\,rc(H)+n\text{-}1. \quad (2)$$

The inequalities (1) and (2) implies the statement. It completes the proof.

**Theorem 1.** *Let $P_n$ and $P_m$ be paths of order $n$ and $m$, $n, m\geq 2$. The rainbow connection number of the comb product of $P_n$ and $P_m$ is*

$$rc(P_n\rhd P_m)=nm\text{-}1.$$

**Proof.** The graph $P_n\rhd P_m$ is a connected graph with the vertex set $V(P_n\rhd P_m)=\{x_i:1\leq i\leq n\}\cup\{x_{i,j}:1\leq i\leq n, 1\leq j\leq m\text{-}1\}$ and the edge set $E(P_n\rhd P_m)=\{x_ix_{i+1}: 1\leq i\leq n\text{-}1\}\cup\{x_ix_{i,1}:1\leq i\leq n\}\cup\{x_{i,j}x_{i,j+1}:1\leq i\leq n, 1\leq j\leq m\text{-}2\}$. Hence $|V(P_n\rhd P_m)|=nm$, $|E(P_n\rhd P_m)|=nm\text{-}1$ and $diam(P_n\rhd P_m)=n+2m\text{-}3$.

We will prove that $rc(P_n\rhd P_m)\geq nm\text{-}1$, by assuming that $rc(P_n\rhd P_m)<nm\text{-}1$. Suppose we need $(nm\text{-}2)$ colors to have all paths of $P_n\rhd P_m$ as a rainbow path. By definition of comb product, graph $P_n\rhd P_m$ has $n$ copies of subgraph $P_m$, denoted by $(P_m)_i$ for $1\leq i\leq n$ and one copy of subgraph $P_n$ which is called a backbone path (grafting path).

(*a*) We known that $u-v$ path with $u\in V((P_m)_k)$ and

$v \in V((P_m)_j)$ for $k\neq j$, $1\le k,j\le n$ always go through the vertices on $P_n$ such that the rainbow edge color in $P_n$ differs from the rainbow edge color in $(P_m)_i$. Thus, path $P_n$ has $n$-1 colors.

(*b*) Given that every vertex $x_{i,j}$ of subgraph $(P_m)_i$ to a vertex $x_i$, of $P_n$ has only one path, namely $x_{i,m-1}$-$x_{i,m-2}$-…-$x_{i,1}$-$x_i$. Thus, each subgraph of path $(P_m)_i$ for $1\le i\le n$ has $m$-1 colors.

(*c*) Since there are at least two same colors of an edge $e\in E((P_m)_k)$ with an edge $e^*\in E((P_m)_j)$ for $k\neq j$, $1\le k,j\le n$, $c(e)=c(e^*)$, the rainbow color of subgraph $(P_m)_i$ will have $(n-1)(m-1)+(m-2)=nm-n-1$ colors.

(*d*) We can take a rainbow path between the vertex $u\in V((P_m)_k)$ and vertex $v\in V((P_m)_i)$ for $k\neq j$, $1\le k,j\le n$ namely

$$\underbrace{x_{i,s}-\cdots-x_{i,1}}_{V((P_m)_k)}-x_1-\cdots-x_l-\underbrace{x_{i,1}-\cdots-x_{i,r}}_{V((P_m)_j)}$$

$$\approx \underbrace{e}_{E((P_m)_k)} - e' - \underbrace{e^*}_{E((P_m)_j)}$$

for $1\le s$, $r\le m$-1 and $1\le l\le n$. Thus, there will be at least two edges in the rainbow path $u-v$ with $u\in V((P_m)_k)$ and $v\in V((P_m)_i)$ having the same colors, namely $c(e)=c(e^*)$. It is a contradiction. Thus, from points (b) and (c), we get $nm-n-1+n-1=nm-2$ colors but it implies that there are at least two edges that have the same color on the rainbow path based on the point (d). Thus, we need at least $nm$-1 colors to color a path of graph $P_n \rhd P_m$ such that all its paths are a rainbow path. We can conclude that the lower bound of the rainbow connection number of $P_n \rhd P_m$ is $rc(P_n \rhd P_m)\ge nm$-1. Furthermore, we will prove that the upper bound of the rainbow connection number of $P_n \rhd P_m$ is $rc(P_n \rhd P_m)\le nm$-1. Define the edge coloring $c$: $E(P_n \rhd P_m)\rightarrow\{1,2,\ldots,nm$-$1\}$ as follows

$$c(e)=\begin{cases} i, \quad e=x_ix_{i+1}; 1\le i\le n-1 \\ n+(i-1)(m-1), \quad e=x_ix_{i,j}; 1\le i\le n,\ j=1 \\ n+(i-1)(m-1)+j, \quad e=x_{i,j}x_{i,j+1}; 1\le i\le n, \\ \qquad\qquad 1\le j\le m-2. \end{cases}$$

It gives a rainbow path from any vertex to other vertices on edge coloring $c$: $E(P_n \rhd P_m)\rightarrow\{1,2,\ldots,nm$-$1\}$. Thus, the upper bound of rainbow connection number of $P_n \rhd P_m$ is $rc(P_n \rhd P_m)\le nm$-1. It concludes that $rc(P_n \rhd P_m)=nm$-1.

**Theorem 2.** *Let $P_n$ be a path of order $n$, $n\ge2$, and $K_m$*

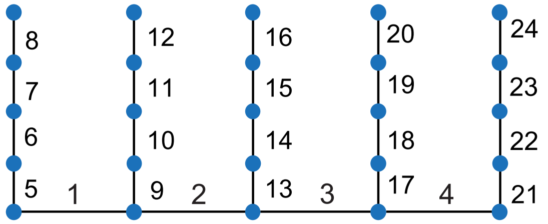


**Fig. 1:** *Example of rainbow connection of $P_5 \rhd P_5$.*

*be a complete graph of order $m$, $m\ge3$. The rainbow connection number of the comb product between $P_n$ and $K_m$ is*

$$rc(P_n \rhd K_m)=2n-1.$$

**Proof.** The graph $P_n \rhd K_m$ is a connected graph with the vertex set $V(P_n \rhd K_m)=\{x_{i,j}: 1\le i\le n,\ 1\le i\le m\}$ and the edge set $E(P_n \rhd K_m)=\{x_{i,1}x_{i+1,1}: 1\le i\le n-1\} \cup \{x_{i,j}x_{i,j+k}: 1\le i\le n,\ 1\le j\le m-1,\ 1\le k\le m-j\}$. Hence $|V(P_n \rhd K_m)|=nm$, $|E(P_n \rhd K_m)|=nm(m-1)/2+n-1$ and $diam(P_n \rhd K_m)=n+1$.

We will prove that $rc(P_n \rhd K_m)\ge 2n-1$, by assuming that $rc(P_n \rhd K_m)<2n-1$. Suppose we need $(2n-2)$ colors to have all paths of $P_n \rhd K_m$ as a rainbow path. By definition of comb product $P_n \rhd K_m$, it will have $n$ copies of subgraph $K_m$, denoted by $(K_m)_i$ for $1\le i\le n$ and one subgraph $P_n$ which is called a backbone path (grafting path).

(*a*) The path $u$–$v$ with $u\in V((K_m)_k)$ and $v\in V((K_m)_j)$ for $k\neq j$, $1\le k,j\le n$, always go through the vertices on $P_n$ such that the rainbow edge color in $P_n$ differs with that edge rainbow path in $(K_m)_i$. Thus, path $P_n$ has $n$-1 rainbow colors.

(*b*) Each path from $x_{i,j}$ of subgraph $(K_m)_i$ to a vertex $x_{i,1}$ of $P_n$ has only one path, namely $x_{i,j}$-$x_{i,1}$ (since the diameter of $(K_m)_i$ is 1). Thus, each complete subgraph $(K_m)_i$ for $1\le i\le n$ has one color.

(*c*) Since there are at least two same colors of an edge $e\in E((K_m)_k)$ with an edge $e^*\in E((K_m)_j)$ for $k\neq j$, $1\le k,j\le n$, $c(e)=c(e^*)$, the rainbow color of subgraph $(K_m)_i$ will have $n$-1 colors.

(*d*) Suppose we take the rainbow path of vertex $u\in V((K_m)_k)$ and vertex $v\in V((K_m)_j)$ for $k\neq j$, $1\le k,j\le n$ namely

$$\underbrace{x_{k,s}}_{V((K_m)_k)}-x_{k,1}-\cdots-x_{j,1}-\underbrace{x_{j,r}}_{V((K_m)_j)}$$

$$\approx \underbrace{e}_{E((K_m)_k)} - e' - \underbrace{e^*}_{E((K_m)_j)}$$

for $1\le s$, $r\le m$ and $1\le k,j\le n$, thus there are at least two edges of rainbow path $u$–$v$ with $u\in V((K_m)_k)$

and $v \in V((K_m)_i)$ having the same colors, namely $c(e)=c(e^*)$. It is a contradiction.

Thus, by points (b) and (c), we get $n$ -1 + $n$ -1 = 2$n$-2 colors, but it implies that there are at least two edges having the same color on its rainbow path based on the point (d). Thus, we can not avoid to have 2$n$-1 colors to color a path of graph $P_n \rhd K_m$ such that all its paths are a rainbow path. We can conclude that the lower bound of the rainbow connection number of $P_n \rhd K_m$ is $rc(P_n \rhd K_m) \geq 2n-1$. Furthermore, we will prove that the upper bound of the rainbow connection number of $P_n \rhd K_m$ is $rc(P_n \rhd K_m) \leq 2n-1$. Define the edge coloring $c: E(P_n \rhd K_m) \to \{1,2,\ldots, 2n-1\}$ as follows

$$c(e)=\begin{cases} i, & e = x_{i,1}x_{i+1,1};\, 1 \leq i \leq n-1 \\ n-1+i, & e = x_{i,j}x_{i,j+k};\, 1 \leq i \leq n,\ 1 \leq j \leq m-1, \\ & 1 \leq k \leq m-j. \end{cases}$$

It gives a rainbow path from any vertex to other vertices on edge coloring $c: E(P_n \rhd K_m) \to \{1,2,\ldots,2n-1\}$. Thus, the upper bound of rainbow connection number of $P_n \rhd K_m$ is $rc(P_n \rhd K_m) \leq 2n-1$. It concludes that $rc(P_n \rhd P_m)=2n-1$.

**Theorem 3.** *Let $P_n$ be a path of order $n$, $n \geq 2$, and $S_m$ be a star of order $m+1$, $m \geq 1$. The rainbow connection number of the comb product of $P_n$ and $S_m$ is*

$$rc(P_n \rhd S_m)=mn+n-1.$$

**Proof.** The graph $P_n \rhd S_m$ is a connected graph with the vertex set $V(P_n \rhd S_m)=\{x_i: 1 \leq i \leq n\} \cup \{x_{i,j}: 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set $E(P_n \rhd S_m)=\{x_i x_{i+1}: 1 \leq i \leq n-1\} \cup \{x_i x_{i,j}: 1 \leq i \leq n,\ 1 \leq j \leq m\}$. Hence $|V(P_n \rhd S_m)|=nm+n$, $|E(P_n \rhd S_m)|=nm+n-1$ and $diam(P_n \rhd S_m)=n+1$.

The grafting vertex of star $S_m$ is a vertex of degree one or pendant vertex. We will prove the lower bound of $rc(P_n \rhd S_m) \geq mn+n-1$, by assuming that $rc(P_n \rhd S_m) < mn+n-1$. Suppose we need $(mn+n-2)$ colors to have all paths of $P_n \rhd S_m$ as a rainbow path. Based on the definition of the comb product $P_n \rhd S_m$, it will has $n$ copies of subgraph $S_m$, denoted by $(S_m)_i$ for $1 \leq i \leq n$ and one subgraph $P_n$ which is called a backbone path (grafting path).

(*a*) We know that $u - v$ with $u \in V((S_m)_k)$ and $v \in V((S_m)_j)$ for $k \neq j$, $1 \leq k,j \leq n$ always go through the vertex in path $P_n$ thus the edge rainbow color of path $P_n$ differs the edge rainbow color of path in $(S_m)_i$. Thus, path $P_n$ has $n$-1 colors to be all rainbow path.

(*b*) Since the diameter of $(S_m)_i$ is 2, each path from vertex $x_{i,j}$ of subgraph $(S_m)_i$ to vertex $x_{i,1}$ in $P_n$ have only one path namely $x_{i,k} - x_{i,1}$. Thus, each subgraph of star $(S_m)_i$ for $1 \leq i \leq n$ has $m$ colors which induce a rainbow coloring of star graph.

(*c*) Since there are at least two same colors of an edge $e \in E((S_m)_k)$ with an edge $e^* \in E((S_m)_j)$ for $k \neq j$, $1 \leq k,j \leq n$, $c(e)=c(e^*)$. The rainbow color of $(S_m)_i$ will have $m(n-1)+m-1=nm-1$ colors.

(*d*) Suppose we take the rainbow path of vertex $u \in V((S_m)_k)$ and $v \in V((S_m)_i)$ for $k \neq j$, $1 \leq k,j \leq n$ namely

$$\underbrace{x_{k,s}}_{V((S_m)_k)} - x_k - \cdots - x_j - \underbrace{x_{j,r}}_{V((S_m)_j)}$$
$$\approx \underbrace{e}_{E((S_m)_k)} - e' - \underbrace{e^*}_{E((S_m)_j)}$$

for $1 \leq s$, $r \leq m$ and $1 \leq k,j \leq n$, there are at least two edges of rainbow path $u - v$ with $u \in V((S_m)_k)$ and $v \in V((S_m)_j)$ having the same colors, namely $c(e)=c(e^*)$. It is a contradiction.

Based on the points (b) and (c) we have $nm$-1+$n$-1=$nm+n$-2 colors but it implies that there are at least two edges having the same color in the rainbow path based on the point (d). Thus, we can not avoid to have at least $nm+n$-1 colors to make all paths of $P_n \rhd S_m$ as a rainbow path. We conlude that the lower bound of the rainbow connection number of $P_n \rhd S_m$ is $rc(P_n \rhd S_m) \geq nm+n-1$. Furthermore, we will prove the upper bound of the rainbow connection number of $P_n \rhd S_m$ that is $rc(P_n \rhd S_m) \leq nm+n-1$. Define the edge coloring $c: E(P_n \rhd S_m) \to \{1,2,3,\ldots,nm+n-1\}$ as follows.

$$c(e)=\begin{cases} i, & e = x_i x_{i+1};\, 1 \leq i \leq n \\ nj-1+i, & e = x_i x_{i,j};\, 1 \leq i \leq n,\ 1 \leq j \leq m. \end{cases}$$

It gives a rainbow path from any vertex to other vertices on edge coloring $c: E(P_n \rhd S_m) \to \{1,2,\ldots, nm+n-1\}$. Thus, the upper bound of rainbow connection number of $P_n \rhd S_m$ is $rc(P_n \rhd S_m) \leq nm+n-1$. Therefore, $rc(P_n \rhd S_m)=nm+n-1$.

**Theorem 4.** *Let $P_n$ be a path of order $n$, $n \geq 2$, and $C_m$ be a cycle of order $m$, $m \geq 3$. The rainbow connection number of the comb product of $P_n$ and $C_m$ is*

$$rc(P_n \rhd C_m)=m+n-1.$$

**Proof.** The graph $P_n \rhd C_m$ is a connected graph with the vertex set $V(P_n \rhd C_m)=x_{i,j}: 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and the edge set $E(P_n \rhd C_m)=\{x_{i,1}x_{i+1,1}: 1 \leq i \leq n-1\} \cup \{x_{i,j}x_{i,j+1},\ x_{i,m}x_{i,1}: 1 \leq i \leq n,\ 1 \leq j \leq m-1\}$. Hence we get $|V(P_n \triangleright C_m)|=nm$, $|E(P_n \triangleright C_m)|=nm+n$-1 and $diam(P_n \triangleright C_m)= 2\lfloor m/2 \rfloor + n - 1$.

Based on Proposition 1, we have that $rc(P_n \triangleright C_m) \geq diam(P_n \triangleright C_m)= 2\lfloor m/2 \rfloor + n - 1$. If $m$ is even then $2\lfloor m/2 \rfloor + n - 1.=2(m/2)+ n$-1$=m+n$-1 and if $m$ is odd then $2\lfloor m/2 \rfloor + n - 1 =2((m$-1$)/2)+n$-1$=m+n$-2. Hence $rc(P_n \triangleright C_m) \geq$ max $\{m+n$-1$,m+n$-2$\}=m+n$-1.

In order to show that $rc(P_n \triangleright C_m) \leq m+n$-1, we define an edge coloring $c: E(P_n \triangleright C_m) \rightarrow \{1,2,\ldots, n+m$-1$\}$ of $P_n \triangleright C_m$ in the following way:

$$c(e)=\begin{cases} i, & e=x_{i,1}x_{i+1,1}; 1\leq i\leq n-1 \\ n-1+j, & e=x_{i,j}x_{i,j+1}; 1\leq i\leq n, 1\leq j\leq m-1 \\ n-1+m, & e=x_{i,1}x_{i,m}; 1\leq i\leq n. \end{cases}$$

The edge coloring gives a rainbow path between every two vertices of $P_n \triangleright C_m$. Therefore, $rc(P_n \triangleright C_m) \leq m+n$-1. Hence $rc(P_n \triangleright C_m)=m+n$-1 and it completes the proof.

As an illustration, we give Figure 2.

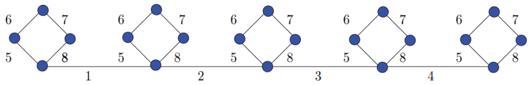


Fig. 2: *Example of rainbow connection of $P_5 \triangleright C_4$.*

**Theorem 5.** *Let $P_n$ be a path of order $n$, $n \geq 2$, and $F_m$ be a fan of order $m+1$, $m \geq 2$. The rainbow connection number of the comb product between $P_n$ and $F_m$ is* $rc(P_n \triangleright F_m) \leq m+n$-1.

**Proof.** Let o be the central vertex of the fan $F_m$. Then the graph $P_n \triangleright F_m$ is a connected graph with the vertex set $V(P_n \triangleright F_m)=\{x_i: 1\leq i\leq n\} \cup \{y_{i,j}: 1\leq i\leq n, 1\leq j\leq m\}$ and the edge set $E(P_n \triangleright F_m)=\{x_i x_{i+1}: 1\leq i\leq n$-1$\} \cup \{x_i y_{i,j}: 1\leq i\leq n, 1\leq j\leq m\} \cup \{y_{i,j} y_{i,j+1}: 1\leq i\leq n, 1\leq j\leq m$-1$\}$. Hence $|V(P_n \triangleright F_m)|=n(m+1)$, $|E(P_n \triangleright F_m)|=2nm$-1 and $diam(P_n \triangleright F_m)=n+1$.

Now, we prove that $rc(P_n \triangleright F_m) \leq m+n$-1 by defining an edge coloring $c$ as follows:

$$c(e)=\begin{cases} i, & e=x_i x_{i+1}; 1\leq i\leq n-1 \\ n-1+j, & e=x_i y_{i,j}; 1\leq i\leq n, 1\leq j\leq m \\ n+m-1-j, & e=y_{i,j}y_{i,j+1}; 1\leq i\leq n, 1\leq j\leq m-1. \end{cases}$$

It is easy to see that the edge coloring $c: E(P_n \triangleright F_m) \rightarrow \{1,2,\ldots,m+n$-1$\}$ is the desired coloring with at most $m+n$-1 colors. It proves that $rc(P_n \triangleright F_m) \leq m+n$-1.

The corona product of a graph $G$ with a graph $H$, denoted by $G \odot H$, is a graph obtained by taking one copy of an $n$-vertex graph $G$ and $n$ copies $H_1, H_2, \ldots, H_n$ of $H$ and then joining the $i$-th vertex of the graph $G$ to every vertex in $H_i$. The graph $Bt_m \cong P_1 \odot S_m$ where $S_m$ is the star of order $m+1$, we call the triangular book graph.

**Theorem 6.** *Let $P_n$ be a path of order $n$, $n \geq 2$, and $Bt_m$ be a triangular book graph of order $m+2$, $m \geq 1$. The rainbow connection number of the comb product between $P_n$ and $Bt_m$ is*

$$rc(P_n \triangleright Bt_m)=n+2.$$

**Proof.** Let $o$ be the vertex of $P_1$ in the triangular book $Bt_m$. The graph $P_n \triangleright Bt_m$ is a connected graph with the vertex set $V(P_n \triangleright Bt_m)=\{a_i, b_i: 1\leq i\leq n\} \cup \{y_{i,j}: 1\leq i\leq n, 1\leq j\leq m\}$ and the edge set $E(P_n \triangleright Bt_m)=\{a_i a_{i+1}: 1\leq i\leq n$-1$\} \cup \{b_i y_{i,j}: 1\leq i\leq n, 1\leq j\leq m\} \cup \{a_i y_{i,j}: 1\leq i\leq n, 1\leq j\leq m\} \cup \{a_i b_i: 1\leq i\leq n\}$. Hence $|V(P_n \triangleright Bt_m)|=n(m+2)$, $|E(P_n \triangleright Bt_m)|=2n(m+1)$-1 and $diam(P_n \triangleright Bt_m)=n+1$.

Based on Proposition 1, we have

$$rc(P_n \triangleright Bt_m) \geq diam(P_n \triangleright Bt_m)=n+1.$$

Now, we define an edge coloring c of the graph $P_n \triangleright Bt_m$ as follows:

$$c(e)=\begin{cases} i, & e=a_i a_{i+1}; 1\leq i\leq n-1 \\ n, & e=a_i y_{i,j}; 1\leq i\leq n, 1\leq j\leq m \\ n+2, & e=b_i y_{i,j}; 1\leq i\leq n, 1\leq j\leq m \\ n+1, & e=a_i b_i; 1\leq i\leq n. \end{cases}$$

It is easy to see that the graph $P_n \triangleright Bt_m$ is rainbow connected under the edge coloring $c: E(P_n \triangleright Bt_m) \rightarrow \{1,2,\ldots,n+2\}$. Therefore, $rc(P_n \triangleright Bt_m) \leq n+2$.

Now, we suppose that $rc(P_n \triangleright Bt_m)=n+1$. However, there is the path between the vertices $y_{1,j}$ and $y_{n,j}$, $1\leq j\leq m$, that is not rainbow path because at least $n+2$ colors are necessary. This proves that $rc(P_n \triangleright Bt_m)=n+2$.

As an illustration, we give Figure 3.

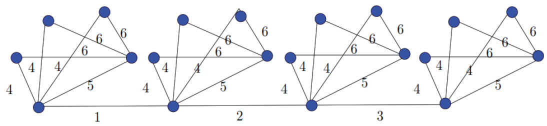


Fig. 3: *Example of rainbow connection of $P_4 \triangleright Bt_3$.*

## 3. Strong Rainbow Coloring

Secondly, we will show the results on the strong rainbow coloring of graphs.

**Lemma 2.** *Let $P_n$ be a path of order $n$, and $H$ be any connected graph. For comb product of $P_n$ and $H$ holds*

$$diam(P_n \rhd H) \le src(P_n \rhd H) \le n \,.\, src(H)+n-1.$$

**Proof.** Let $H$ be any graph of order $p$ and size $q$ and $P_n$ be a path of order $n$. From Proposition 1 it follows

$$src(P_n \rhd H) \ge diam(P_n \rhd H). \qquad (3)$$

Based on the definition of the comb product $P_n \rhd H$, the graph $H$ is copied $n$ times, and the graph $P_n \rhd H$ is strongly rainbow connected if there exists a rainbow $u-v$ geodesic for any two vertices $u$ and $v$ in $P_n \rhd H$. Thus we have the upper bound for the strong rainbow connection number as follows

$$src(P_n \rhd H) \le n \,.\, src(H)+src(P_n)=n \,.\, src(H)+n-1. \qquad (4)$$

The inequalities (3) and (4) imply that $diam(P_n \rhd H) \le src(P_n \rhd H) \le n.src(H)+n-1$. It completes the proof.

**Theorem 7.** *Let $P_n$ be a path of order $n$, $n \ge 2$, and $C_m$ be a cycle of order $m$, $m \ge 3$. The strong rainbow connection number of the comb product between $P_n$ and $C_m$ is*

$$src(P_n \rhd C_m) \leqslant 2\left\lfloor \frac{m}{2} \right\rfloor + n-1.$$

**Proof.** According to Proposition 1 we get

$$src(P_n \rhd C_m) \ge diam(P_n \rhd C_m) = 2\left\lfloor \frac{m}{2} \right\rfloor + n-1.$$

With respect to Lemma 2, we have

$$diam(P_n \rhd C_m) \le src(P_n \rhd C_m) \le n.src(C_m)+n-1$$
$$= 2\left\lfloor \frac{m}{2} \right\rfloor + n-1.$$

Now, we define an edge coloring $c$ of the graph $P_n \rhd C_m$ in the following way:

$$c(e)=\begin{cases} i, \quad e=x_{i,1}x_{i+1,1}; 1\le i\le n-1 \\ n-1+j+(i-1)\lfloor m/2 \rfloor, \\ \qquad e=x_{i,j}x_{i,j+1}; 1\le i\le n, 1\le j\le \lfloor m/2 \rfloor \\ n-1+j-\lfloor m/2 \rfloor+(i-1)\lfloor m/2 \rfloor, \\ \qquad e=x_{i,j}x_{i,j+1}; 1\le i\le n, \lfloor m/2 \rfloor+1\le j\le m-1 \\ n-1+i\lfloor m/2 \rfloor, \quad e=x_{i,1}x_{i,m}; 1\le i\le n \\ 1, \quad e=x_{i,\lfloor m/2 \rfloor}x_{i,\lfloor m/2 \rfloor+1}; 1\le i\le n. \end{cases}$$

It is not difficult to see that the graph $P_n \rhd C_m$ is the strong rainbow connected under the edge coloring $c$: $E(P_n \rhd C_m) \rightarrow \{1,2,\ldots, n\lfloor m/2 \rfloor + n-1\}$. It proves that $src(P_n \rhd C_m) \le n\lfloor m/2 \rfloor + n-1$.

Figure 4 gives an example of strong rainbow connection of $P_5 \rhd C_4$.

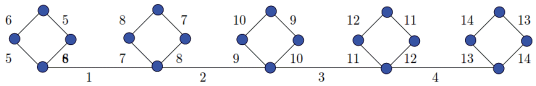


**Fig. 4:** *Example of rainbow connection of $P_5 \rhd C_4$.*

**Theorem 8.** *Let $P_n$ be a path of order $n$, $n \ge 2$, and $F_m$ be a fan of order $m+1$, $m \ge 2$. The strong rainbow connection number of the comb product between $P_n$ and $F_m$ is*

$$src(P_n \rhd F_m) \le mn+n-1.$$

**Proof.** Define an edge coloring $c$ of the graph $P_n \rhd F_m$ in the following way:

$$c(e)=\begin{cases} i, \quad e=x_i x_{i+1}; 1\le i\le n-1 \\ n-1+j+(i-1)m, \quad e=x_i y_{i,j}; 1\le i\le n, 1\le j\le m \\ 1, \quad e=y_{i,j}y_{i,j+1}; 1\le i\le n, 1\le j\le m-1. \end{cases}$$

We can see that the graph $P_n \rhd F_m$ is rainbow connected under the previous edge coloring $c$ which uses at most $mn+n-1$ colors. Thus, the existence of the coloring $c$: $E(P_n \rhd F_m) \rightarrow \{1,2,\ldots, mn+n-1\}$ proves that $src(P_n \rhd F_m) \le mn+n-1$.

An example of strong rainbow connection of $P_n \rhd F_m$ is depicted in Figure 5.

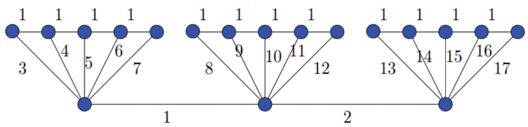


**Fig. 5:** *Example of rainbow connection of $P_3 \rhd F_5$.*

**Theorem 9.** *Let $P_n$ be a path of order $n$, $n \ge 2$, and $Bt_m$ be a triangular book graph of order $m+2$, $m \ge 1$. The strong rainbow connection number of the comb product between $P_n$ and $Bt_m$ is*

$$src(P_n \rhd Bt_m) \leqslant mn+n-1.$$

**Proof.** Let us define an edge coloring $c$: $E(P_n \rhd Bt_m) \rightarrow \{1,2,\ldots, mn+n-1\}$ as follows:

$$c(e)=\begin{cases} i, \quad e=a_i a_{i+1}; 1\le i\le n-1 \\ n+m(i-1)+j-1, \quad e=a_i y_{i,j}; 1\le i\le n, 1\le j\le m \\ n+m(i-1)+j-1 \quad e=b_i y_{i,j}; 1\le i\le n, 1\le j\le m \\ n+m(i-1), \quad e=a_i b_i; 1\le i\le n. \end{cases}$$

It is not difficult to verify that the edge coloring $c$ gives a rainbow $u$-$v$ geodesic for every two vertices $u$ and $v$ of triangular book $P_n \rhd Bt_m$. Since the coloring $c$ uses at most $nm+n$-1 colors then $src(P_n \rhd Bt_m) \le nm+n$-1.

As an illustration, we give Figure 6.

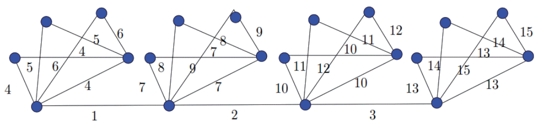

Fig. 6: *Example of rainbow connection of $P_4 \rhd Bt_3$.*

## 4. Concluding Remarks

In this paper, we have determined the exact values of the rainbow connection number of the comb product between a path $P_n$ and a cycle $C_m$, and between a path $P_n$ and a triangular book graph $Bt_m$. We obtained only the upper bound of the rainbow connection number of the comb product between path $P_n$ and fan $F_m$. The rest, for the strong rainbow connection number of $P_n \rhd C_m$, $P_n \rhd F_m$ and $P_n \rhd Bt_m$ we have obtained only the upper bounds. Thus we propose the following open problem: "Find the lower bound of the rainbow or strong rainbow connection number of $P_n \rhd C_m$, $P_n \rhd F_m$ and $P_n \rhd Bt_m$".


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## References and Notes


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