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A Construction of *H*-antimagic Graphs

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Abstract: Let $G = (V,E)$ be a finite simple graph with p vertices and q edges. An *edge-covering* of G is a family of subgraphs $H_1,H_2,\cdots,H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i=1,2,\cdots,t$. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an *H-covering*. Such a graph G is called *(a,d)-H-antimagic* if there is a bijection $f\colon V\cup E\rightarrow\{1,2,\cdots,p+q\}$ such that for all subgraphs H' of G isomorphic to H, the sum of the labels of all the edges and vertices belonging to H' constitutes an arithmetic progression with the initial term a and the common difference d. When $f(V)=\{1,2,\cdots,p\}$, then G is said to be *super (a,d)-H-antimagic*; and if $d = 0$ then G is called *H-supermagic*.
We will exhibit an operation on graphs which keeps super *H-antimagic* properties. We use a technique of partitioning sets of integers for the construction of the required labelings.

Keywords: H-covering, (a,d)-H-antimagic graph, super (a,d)-H-antimagic graph, partition of set.

1. Introduction

An *edge-covering* of a finite and simple graph G is a family of subgraphs $H_1,H_2,\cdots,H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i=1,2,\cdots,t$. In this case we say that G admits an *$(H_1,H_2,\cdots,H_t)$-(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an *H-covering*. Suppose that a (p,q)-graph $G=(V,E)$ with p vertices and q edges admits an *H-covering*. The graph G is called *(a,d)-H-antimagic* if there exists a total labeling $f\colon V(G)\cup E(G)\rightarrow\{1,2,\cdots,p+q\}$ such that, for all subgraphs H' of G isomorphic to H, the *H-weights,*

$$wt_f(H')=\sum_{v\in V(H')} f(v)+\sum_{e\in E(H')} f(e),$$

constitute an arithmetic progression a, $a+d$, $a+2d,\cdots$, $a+(t-1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. Moreover, G is said to be *super (a,d)-H-antimagic* if the smallest possible labels appear on the vertices. If G is a (super) *(a,d)-H-antimagic* graph then the corresponding total labeling f is called a *(super) a,d)-H-antimagic labeling*. For $d = 0$, a (super) *(a,d)-H-antimagic* graph is called *H-magic* and *H-supermagic*, respectively.

The H-(super)magic graph was first introduced by Gutiérrez and Lladó in [9]. They proved that some classes of connected graphs are H-supermagic, for example, the stars $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$-supermagic for some h. They also proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Lladó and Moragas [15] investigated C_n-(super)magic graphs and proved that

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wheels, windmills, books and prisms are C_h-magic for some h. Some results on C_n-supermagic labelings of several classes of graphs can be found in [19]. Maryati et al. [17] gave P_h-(super)magic labelings of some trees such as shrubs, subdivision of shrubs and banana tree graphs. Other examples of H-supermagic graphs with different choices of H have been given by Jeyanthi and Selvagopal in [12]. Maryati et al. [18] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_h-supermagic for some c and h.

The (a,d)-H-antimagic labeling was introduced by Inayah et al. [10]. In [11] Inayah et al. investigated the super (a,d)-H-antimagic labelings for some shackles of a connected graph H.

For $H \cong K_2$, (super) (a,d)-H-antimagic labelings are also called simply *(super) (a,d)-edge-antimagic total labelings*. These labelings are the generalization of the edge-magic and super edge-magic labelings that were introduced by Kotzig and Rosa [13] and Enomoto et al. [7], respectively. For further information on (super) edge-magic labelings, see [4, 5, 8, 16].

The (super) (a,d)-H-antimagic labeling is related to a super d-antimagic labeling of type $(1,1,0)$ of a plane graph that is the generalization of a face-magic labeling introduced by Lih [14]. Further information on super d-antimagic labelings can be found in [1, 2, 3, 6].

In this paper we show one operation on graphs which keeps super H-antimagic properties. We use a technique of partitioning sets of integers for a construction of the required labelings.

2. Constructions Using Partitions of Integers

In this section we examine the existence of the super H-antimagic labelings for graphs obtained by one graph operation. The constructions of labelings will be made using partitions of the sets of integers.

Consider the partition $\mathcal{P}_2^n$ of the set of integers $\{1,2,\cdots,2n\}$ into n, $n \geq 2$, couples such that the sums of the numbers in all couples are the same. If $\mathcal{P}_2^n(i)$ denotes the ith couple in the partition $\mathcal{P}_2^n$ then, for example,

$$\mathcal{P}_2^n(i) = \{a_i, b_i\} = \{i, 2n+1-i\},$$

where $i=1,2,\cdots,n$. For the sum of the two numbers in the ith couple we have

$$\sum \mathcal{P}_2^n(i) = a_i + b_i = 2n+1$$

for $i=1,2,\cdots,n$.

A similar idea can be also used for a partition of the set of kn consecutive integers into k-tuples.

Let n, k and i be positive integers. We will consider the partition $\mathcal{P}_k^n$ of the set $\{1,2,\cdots,kn\}$ into n, $n \geq 2$, k-tuples such that the sum of the numbers in the ith k-tuple is always the same and equal to the constant $k(1+kn)/2$, where $i=1,2,\cdots,n$. Using the divisibility we have that if k is odd then n has to be odd too.

Let us consider the partition $\mathcal{P}_3^n$ of the set of integers $\{1,2,\cdots,3n\}$, n odd, into triples such that the ith triple in the partition is defined in the following way

$$\mathcal{P}_3^n(i) = \{a_i, b_i, c_i\} = \begin{cases} \left\{\frac{n+i}{2}, n+\frac{i+1}{2}, 3n+1-i\right\} & \text{for } i \equiv 1 \pmod 2, \\ \left\{\frac{i}{2}, \frac{3n+1+i}{2}, 3n+1-i\right\} & \text{for } i \equiv 0 \pmod 2. \end{cases}$$

It is easy to see that the sum of all numbers in the ith triple is equal to

$$\sum \mathcal{P}_3^n(i) = a_i + b_i + c_i = \frac{3(3n+1)}{2} \quad (1)$$

for $i=1,2,\cdots,n$. Moreover, the minimum of the numbers in every triple is the number from the set $\{1,2,\cdots,n\}$.

Only for the purposes of this paper by the notation $\mathcal{P}_k^n(i) \oplus c$ we mean that the constant c is added to every element of $\mathcal{P}_k^n(i)$.

By $A-B$ we denote the difference of the set B from the set A.

Let G_2^k be a graph obtained from two isomorphic graphs G and G' by connecting corresponding vertices of G and G' with a matching, then subdivide every edge of the matching using k vertices. If a (p,q)-graph G has vertex set $V(G)=\{v_1,v_2,\cdots,v_p\}$ then the graph G_2^k has the vertex set

$$V(G_2^k)=V(G)\cup V(G')\\ \cup\{v_i^j: i=1,2,\cdots,p, j=1,2,\cdots,k\}$$

and the edge set

$$E(G_2^k)=E(G)\cup E(G')\\ \cup\left\{v_iv_i^1, v_i^jv_i^{j+1}, v_i^kv_i': i=1,2,\cdots,p,\\ j=1,2,\cdots,k-1\right\}.$$

The graph G_2^k has $(k+2)p$ vertices and $2q+(k+1)p$ edges.

A useful property for finding H-antimagic labelings is given in the following lemma.

Lemma 1.

Let f be a super (a,d)-H-antimagic labeling of $G=(V,E)$ and let r, s be nonnegative integers. Then the labeling

$$g:V(G)\cup E(G)\to\left\{r+1,r+2,\cdots,r+|V(G)|,\\ r+s+|V(G)|+1,\\ r+s+|V(G)|+2,\cdots,\\ r+s+|V(G)|+|E(G)|\right\}$$

defined such that

$$g(v)=f(v)+r \qquad \text{if } v\in V(G),\\ g(e)=f(e)+r+s \qquad \text{if } e\in E(G)$$

has the property that

$$\left\{wt_g(H): H\subset G\right\}=\left\{b,b+d,\cdots,b+(t-1)d\right\},$$

where t is the number of all subgraphs in G isomorphic to H and b is a positive integer.

Proof.

Let

$$f:V(G)\cup E(G)\to\left\{1,2,\cdots,|V(G)|+|E(G)|\right\}$$

be a super (a,d)-H-antimagic labeling of $G=(V,E)$ and let $H_1,H_2,\cdots,H_t$ be all subgraphs of G isomorphic to H. Thus the set of all H-weights under the labeling f is

$$\left\{wt_f(H_i): i=1,2,\cdots,t\right\}\\ =\left\{a,a+d,\cdots,a+(t-1)d\right\}. \tag{2}$$

For the H-weight of the subgraph H_i, $i=1,2,\cdots,t$, under the labeling g we have

$$\begin{aligned} wt_g(H_i)&=\sum_{v\in V(H_i)}g(v)+\sum_{e\in E(H_i)}g(e)\\ &=\sum_{v\in V(H_i)}\left(f(v)+r\right)\\ &\quad+\sum_{e\in E(H_i)}\left(f(e)+r+s\right)\\ &=\sum_{v\in V(H_i)}f(v)+\sum_{e\in E(H_i)}f(e)\\ &\quad+r|V(H_i)|+(r+s)|E(H_i)|\\ &=wt_f(H_i)+r|V(H_i)|\\ &\quad+(r+s)|E(H_i)|. \end{aligned}$$

As all subgraphs H_i are isomorphic to H it holds

$$|V(H_i)|=|V(H)|,\\ |E(H_i)|=|E(H)|.$$

Thus $r|V(H_i)|+(r+s)|E(H_i)|$ is a constant for all $i=1,2,\cdots,t$ and

$$wt_g(H_i)=\\ wt_f(H_i)+r|V(H)|+(r+s)|E(H)|. \tag{3}$$

According to the property (3) and by using (2) we get

$$\left\{wt_g(H_i): i=1,2,\cdots,t\right\}\\ =\left\{b,b+d,\cdots,b+(t-1)d\right\},$$

where $b=a+r|V(H)|+(r+s)|E(H)|$.

□

Theorem 2.

Let G be a super (a,d)-H-antimagic graph of odd order containing t subgraphs isomorphic to H and let k be a positive integer, $k \geq 1$. If the graph G_2^k contains exactly t subgraphs isomorphic to H_2^k then the graph G_2^k is super $(b,2d)$-H_2^k-antimagic.

Proof.

Let $f: V(G) \cup E(G) \to \{1,2,\cdots,p+q\}$ be a super (a,d)-H-antimagic labeling of a (p,q)-graph G of odd order and let $H_1, H_2, \cdots, H_t$ be a family of all subgraphs of G isomorphic to H.

Clearly, the set of all H-weights is as follows.

$$\{wt_f(H_l) : l = 1,2,\cdots,t\} = \{a, a+d, \cdots, a+(t-1)d\} \quad (4)$$

and the smallest possible labels $1,2,\cdots,p$ appear on the vertices of G.

Let us consider the labeling g of the vertices and edges of G_2^k defined in the following way.

$$g(v) = f(v) \quad \text{if } v \in V(G),$$

$$g(v') = f(v) + p \quad \text{if } v' \in V(G'),$$

$$g(v_i^1) = \min\{\mathcal{P}_3^p(i)\} \oplus 2p \quad \text{if } i = 1,2,\cdots,p,$$

$$g(v_i^j) = i + (j+1)p \quad \text{if } i = 1,2,\cdots,p, \; j = 2,3,\cdots,k,$$

$$g(e) = f(e) + (k+1)p \quad \text{if } e \in E(G),$$

$$g(e') = f(e) + (k+1)p + q \quad \text{if } e' \in E(G'),$$

$$\{g(v_i v_i^1), g(v_i^k v_i')\} = \mathcal{P}_3^p(i) \oplus ((k+1)p + 2q) - (\min\{\mathcal{P}_3^p(i)\} \oplus ((k+1)p + 2q)) \quad \text{if } i = 1,2,\cdots,p,$$

$$g(v_i^{j-1} v_i^j) = (2k+6)p + 2q + 1 - g(v_i^j) \quad \text{if } i = 1,2,\cdots,p, \; j = 2,3,\cdots,k.$$

It is easy to see that g is a bijection from the vertex set and edge set of G_2^k onto the set $\{1,2,\cdots,(2k+3)p+2q\}$ and the vertices of G_2^k are labeled with the smallest possible labels.

For the H_2^k-weight of the subgraph $(H_l)_2^k$, $l=1,2,\cdots,t$, under the labeling g we get

$$wt_g\left((H_l)_2^k\right) = \sum_{v \in V\left((H_l)_2^k\right)} g(v) + \sum_{e \in E\left((H_l)_2^k\right)} g(e)$$

$$= \left(\sum_{v \in V(H_l)} g(v) + \sum_{v \in V(H_l')} g(v) + \sum_{j=1}^{k} \sum_{i: v_i \in V(H_l)} g(v_i^j) \right) + \left(\sum_{e \in E(H_l)} g(e) + \sum_{e \in E(H_l')} g(e) + \sum_{i: v_i \in V(H_l)} \left(g(v_i v_i^1) + g(v_i^k v_i') \right) + \sum_{j=2}^{k} \sum_{i: v_i \in V(H_l)} g(v_i^{j-1} v_i^j) \right)$$

$$= wt_g(H_l) + wt_g(H_l') + \sum_{i: v_i \in V(H_l)} \left(g(v_i^1) + g(v_i v_i^1) + g(v_i^k v_i') \right) + \sum_{j=2}^{k} \sum_{i: v_i \in V(H_l)} \left(g(v_i^j) + g(v_i^{j-1} v_i^j) \right).$$

According to (3) we get

$$wt_g(H_l) = wt_f(H_l) + (k+1)p|E(H)|,$$

$$wt_g(H_l') = wt_f(H_l) + p|V(H)| + (kp + p + q)|E(H)|.$$

According to (1) it holds

$$\sum_{i: v_i \in V(H_l)} \left(g(v_i^1) + g(v_i v_i^1) + g(v_i^k v_i') \right) = \sum_{i: v_i \in V(H_l)} \Big((\min\{\mathcal{P}_3^p(i)\} \oplus 2p) + \sum (\mathcal{P}_3^p(i) \oplus ((k+1)p + 2q)) - (\min\{\mathcal{P}_3^p(i)\} \oplus ((k+1)p + 2q)) \Big)$$

$$= \sum_{i:v_i \in V(H_l)} \sum \left(\mathcal{P}_3^p(i)\right) + 2\left((k+2)p+2q\right)\left|V(H_l)\right|$$

$$= \frac{3(3p+1)\left|V(H_l)\right|}{2} + 2\left((k+2)p+2q\right)\left|V(H_l)\right|$$

$$= \frac{(4k+17)p+8q+3}{2}\left|V(H_l)\right|$$

$$= \frac{(4k+17)p+8q+3}{2}\left|V(H)\right|.$$

Moreover,

$$\sum_{j=2}^{k} \sum_{i:v_i \in V(H_l)} \left(g(v_i^j) + g(v_i^{j-1}v_i^j)\right)$$

$$= \sum_{j=2}^{k} \sum_{i:v_i \in V(H_l)} \left(g(v_i^j) + (2k+6)p + 2q + 1 - g(v_i^j)\right)$$

$$= \sum_{j=2}^{k} \sum_{i:v_i \in V(H_l)} \left((2k+6)p + 2q + 1\right)$$

$$= \left((2k+6)p+2q+1\right)(k-1)\left|V(H_l)\right|$$

$$= \left((2k+6)p+2q+1\right)(k-1)\left|V(H)\right|.$$

Thus, for $l=1,2,\cdots,t$ we get

$$wt_g\left((H_l)_2^k\right) = 2wt_f(H_l) + A,$$

where

$$A = \left((2k+2)p+q\right)\left|E(H)\right| + \frac{(4k^2+12k+7)p + (4k+4)q + 2k + 1}{2}\left|V(H)\right|.$$

Using (4) we have

$$\left\{wt_g(H_l) : l = 1,2,\cdots,t\right\}$$

$$= \left\{2a + A, 2a + A + 2d, \cdots, 2a + A + 2(t-1)d\right\}.$$

This concludes the proof that the graph G_2^k is super $(2a+A,2d)$-H_2^k-antimagic.

□

3. Conclusion

In this paper we examined the existence of super H_2^k-antimagic labelings for graphs G_2^k obtained from two isomorphic graphs G and G' by joining every couple of corresponding vertices $v \in V(G)$ and $v' \in V(G')$ by a path of length $k+1$.

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5. References and Notes

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