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Stress and Strain Analysis of Curved Beams of Fibre Reinforced Plastic

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BIOGRAPHICAL NOTES

Prof. Dr.-Ing. Peter Heinze is a professor of engineering mechanics at the Hochschule Wismar – University of Applied Sciences, Technology, Business and Design. In 1980 he graduated at the University of Hannover, Department of Civil Engineering, specialisation: the structural mechanics. After years spent in professional engineering positions in various engineering companies he became the Dean of the Faculty of Engineering Sciences at the Hochschule Wismar during years 2004 to 2006 and in the next time period from 2006 to 2009 he was the Head of the Department of Mechanical Engineering/Process- and Environmental Engineering.

KEY WORDS

Highly Curved Beams, Ber Reinforced Plastic, Classical Laminae Theory

ABSTRACT

This paper extends the theory of Leipholz [2] about strongly curved beams to anisotropic ber materials. For this purpose, the classical laminate theory, as proposed by Halpin [1] and others has been applied to curved beams. The laminate is made up of several orthotropic layers with a dened orientation (angle φ), it gives the laminate anisotropic properties. The method presented here can determine stress in and across the ber as a result of internal forces.

INTRODUCTION

Fiber-reinforced plastics found because of high exibility, high specic strength and stiness of an increasingly prevalent. For the modern new material joining techniques such as bonding and rivets or loops have been pushed to the fore. Loops are often seen as one-dimensional but highly curved components, which are composed of several dierently oriented orthotropic layers.

Each layer consists of oriented bers made of glass or carbon, embedded in plastic (epoxy or polyester resin) as the matrix. The ber volume fraction, which increases the stiness and strength is, according to the manufacturing process 20-60 %.

The problem of stress calculation of highly curved beams has been processed already by Leipholz (1969) [2] for isotropic materials. This paper will give an extension to anisotropic materials.

Assumptions and Conditions

The investigation is based on the following assumptions:

- *The beam is loaded only by bending moment and axial force. The transversal bending moment, shear force and torque are not taken into account.*
- *The laminate consists of several layers. The layer consists of directed parallel laments*

surrounded by a matrix.

- *The mechanical behavior of the laminate depends strongly from sequence and angle of laments.*
- *Fiber and matrix are considered as a homogeneous material with an ideal compound. Statements about stress and the adhesion between ber and matrix are not possible.*
- *The generalized Hooke's law for orthotropic material is valid.*
- *Bernoulli's hypothesis is applied.*

With these conditions, a relationship between beam stress resultants and strains will be developed, which allows to determine stress of each layer.

Hooke's Law for Orthotropic Material

Orthotropic material are described in contrast to isotropic materials with four material constants [1]:

E_1 modulus of elasticity in ber direction
E_2 modulus of elasticity perpendicular to ber direction
G_{12} shear modulus of elasticity in plane
υ_{12} major Poisson Ratio
υ_{21} minor Poisson Ratio

Hooke's law describes the relationship of stress and strain, here Hooke's law for plane stress state is given:

$$\underbrace{\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}}_{\vec{\sigma}} = \underbrace{\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{21} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \gamma_{12}\end{bmatrix}}_{\vec{\varepsilon}}, \quad (1)$$

with

$$Q_{11} = \frac{E_{11}}{1 - \upsilon_{12}\upsilon_{21}},$$

$$Q_{12} = \frac{\upsilon_{12}E_{22}}{1 - \upsilon_{12}\upsilon_{21}} = Q_{21} = \upsilon_{21}\frac{E_{11}}{1 - \upsilon_{12}\upsilon_{21}},$$

$$Q_{22} = \frac{E_{22}}{1 - \upsilon_{12}\upsilon_{21}},$$

$$Q_{66} = G_{12}.$$

The matrix **Q** is called the stiness matrix, the indices 1 and 2 indicate here that it is related to the material axes. The vectors $\vec{\sigma}$ and $\vec{\varepsilon}$ are the stress respectively the strain vector both related to material axis.
The relationship of strain and stress is given in equation (2). The matrix **S** is called compliance matrix.
Of course the relationship is $\mathbf{Q}^{-1} = \mathbf{S}$ is valid.

$$\underbrace{\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \gamma_{12}\end{bmatrix}}_{\vec{\varepsilon}} = \underbrace{\begin{bmatrix}S_{11} & S_{12} & 0\\ S_{21} & S_{22} & 0\\ 0 & 0 & S_{66}\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{bmatrix}}_{\vec{\sigma}}, \quad (2)$$

With

$$S_{11} = \frac{1}{E_1}; \quad S_{12} = -\frac{\upsilon_{12}}{E1}; \quad S_{21} = -\frac{\upsilon_{21}}{E_2} = S_{12};$$

$$S_{22} = \frac{1}{E_2}; \quad S_{66} = \frac{1}{G_{12}}$$

Stiness matrix as well as compliance matrix are symmetric, their structure indicates that no coupling between normal stress and shear strain occurs.

Transformations

To study the behavior of the entire laminate, both the stiffness matrix and the compliance matrix of a layer are transformed in a common coordinate system.
This common coordinate system is called the x-y-system. The angle between the common x-axis and the ber direction of the i^{th} layer is φ_i.
The stress vector is transformed by equation (3). The superscript $\bar{\sigma}$ denotes the x-y-system.

$$\underbrace{\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}}_{\vec{\sigma}} = T_\sigma \underbrace{\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}}_{\vec{\bar{\sigma}}}. \quad (3)$$

Since we use engineering shear strain instead of tensor shear strain it is essential for $\boldsymbol{\varepsilon}$ a second transformation relationship

$$\underbrace{\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \gamma_{12}\end{bmatrix}}_{\vec{\varepsilon}} = T_\varepsilon \underbrace{\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix}}_{\vec{\bar{\varepsilon}}}, \quad (4)$$

with

$$T_\sigma = \begin{bmatrix}c^2 & s^2 & 2c\,s\\ s^2 & c^2 & -2\,c\,s\\ -c\,s & c\,s & c^2 - s^2\end{bmatrix}, \quad (5)$$

$$T_\sigma^{-1} = \begin{bmatrix}c^2 & s^2 & -2c\,s\\ s^2 & c^2 & 2\,c\,s\\ c\,s & -c\,s & c^2 - s^2\end{bmatrix}, \quad (6)$$

and

$$T_\varepsilon = \begin{bmatrix} c^2 & s^2 & c\,s \\ s^2 & c^2 & -c\,s \\ -2\,c\,s & 2\,c\,s & c^2 - s^2 \end{bmatrix}, \tag{7}$$

$$T_\varepsilon^{-1} = \begin{bmatrix} c^2 & s^2 & -c\,s \\ s^2 & c^2 & c\,s \\ 2\,c\,s & -2\,c\,s & c^2 - s^2 \end{bmatrix}, \tag{8}$$

with:

$$c = cos\varphi \text{ and } s = sin\varphi. \tag{9}$$

With equation (3) to (8) the transformation of stiffness and compliance matrix will be performed:

$$\bar{\mathbf{Q}} = \mathbf{T}_\sigma^{-1}\,\mathbf{Q}\,\mathbf{T}_\varepsilon \tag{10}$$

and

$$\bar{\mathbf{S}} = \mathbf{T}_\varepsilon^{-1}\,\mathbf{S}\,\mathbf{T}_\sigma. \tag{11}$$

The transformed stiness and compliance matrices are still symmetric but fully populated. That is why a coupling of normal stress and shear strain can be observed.

Stress - Strain Relation of a Plane Beam

The beam theory assumes that stresses perpendicular to the axis of the beam in comparison to those in beam's axis direction are negligible, therefor the stress vector is given by:

$$\vec{\bar{\sigma}} = [\sigma_x \;\; 0 \;\; 0]^T. \tag{12}$$

The strain induced by a unit stress $\sigma_x = 1$ is the first column of compliance matrix:

$$\vec{\bar{\varepsilon}}_u = \bar{S}[1 \;\; 0 \;\; 0]^T. \tag{13}$$

The strain ε_x induced by a unit stress σ_x is the element in the first column and the first row of the compliance matrix $\bar{\mathbf{S}}$ in global coordinate system. Therefore the inverse of the matrix element $\bar{\mathbf{S}}_{11}$ can be regarded as a generalized modulus of elasticity in x-direction of the i^{th} layer.

$$E_x^i = \frac{\vec{\bar{\sigma}}_u(1)}{\vec{\bar{\varepsilon}}_u(1)} = \frac{1}{\bar{S}_{1,1}}. \tag{14}$$

The i denotes that it is a stress for unit strain for the i^{th} layer.

LAMINATED CURVED BEAM

It is known, that in highly curved beam loaded by pure bending the neutral axis does not coincides with the centroid. Therefore the first question is to find out the zero strain position. For this purpose in Fig. 1 a differential piece of the beam is shown.

The designations are:

R_s - distance of center of curvature to centroid,

R - distance of center of curvature to neutral axis,

ρ - coordinate starting at center of curvature,

$\bar{z}$ - coordinate in ρ direction, staring at neutral axis,

$\Delta d\varphi$ - increase of curvature due to bending moment,

b - width of beam.

Coordinate System

The origin of the coordinate ρ is the center of curvature. R is the distance from the center of curvature to the neutral axis. The origin of the local beam coordinate system is the neutral axis of a pure bending loaded beam. $\bar{z}$ is directed outwards, x is following the beam axis. The fiber angle φ is measured from the x-axis.

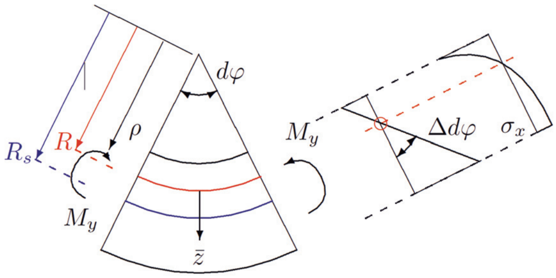

Fig. 1 *Bending of highly curved beam*

Determination of Position of Neutral Axis

Starting point is the assumption that a pure bending moment and no axial force is acting. The axial force is:

$$N = \int \sigma_x dA = 0. \tag{15}$$

Due to Bernoulli's hypothesis an assumption about the displacement u_x can be made:

$$u_x = \Delta d\varphi\,\bar{z}, \tag{16}$$

with

$$\bar{z} = \rho - R. \tag{17}$$

The strain ε is:

$$\varepsilon_x = \frac{\Delta d\varphi\ \bar{z}}{d\varphi\ \rho},$$

so that the stress becomes:

$$\sigma_x^i = \varepsilon_x^i E_x^i = \frac{\Delta d\varphi\ \bar{z}}{d\varphi\ \rho} E_x^i.$$

The axial force of one layer is the integral over the layer area, the axial force of the laminate the summation over all layers:

$$N = \sum_i \int_A \frac{\Delta d\varphi(\rho - R)}{d\varphi\ \rho} E_x^i dA. \tag{18}$$

The term $\frac{\Delta d\varphi}{d\varphi}$ and remains constant and is shifted in front of the summation, the term E_x^i remains constant for one layer and is shifted in front of the integral:

$$N = \frac{\Delta d\varphi}{d\varphi} \underbrace{\sum_i E_x^i \int_A \frac{\rho - R}{\rho} dA}_{0} = 0. \tag{19}$$

The summation is set to zero;

$$\sum_i E_x^i \int_A \frac{\rho - R}{\rho} dA = 0$$

$$= \sum_i E_x^i \left[\int_A \frac{\rho}{\rho} dA - \int_A \frac{R}{\rho} dA \right].$$

The first integral gives the cross section area A_i of the i^{th} layer, the second leads to

$$\int_A \frac{R}{\rho} dA = b\ R \left[\ln \rho \right]_{R_i}^{R_{i+1}},$$

with R_i the borders of layer i. This leads to the equation to determine R:

$$\sum_i E_x^i A_i - b\ R \sum_i E_x^i \left[\ln \rho \right]_{R_i}^{R_{i+1}} = 0. \tag{20}$$

The radius R of neutral axis gives:

$$R = \frac{\sum_i E_x^i A_i}{b \sum_i E_x^i \left[\ln \rho \right]_{R_i}^{R_{i+1}}}. \tag{21}$$

STRESS RESULTANTS AS INTEGRAL OF STRESS

Axial Force

The axial force of a beam is

$$N = \int \sigma_x dA, \tag{22}$$

using Hooke's law in connection with equation (14) leads to

$$N = \sum_i E_x^i \int \varepsilon_x dA. \tag{23}$$

Strain is simply spoken

$$\varepsilon = \frac{\text{new length - old length}}{\text{old length}}.$$

The strain $\boldsymbol{\varepsilon}_x$ is composed of a constant share $\boldsymbol{\varepsilon}_{x0}$ and a variable portion due to the increase in curvature $\boldsymbol{\kappa}$. For this second part it is taken into account that also the "old length" is variable with $\bar{z}$. Taking this into account we obtain

$$\varepsilon_x = \varepsilon_{xo} \frac{R}{\rho} + \frac{\Delta d\varphi \bar{z}}{\frac{\rho}{R}} = \varepsilon_{xo} \frac{R}{\rho} + R \frac{\Delta d\varphi \bar{z}}{\rho}. \tag{24}$$

Introducing equation (24) in equation (23) leads to

$$N = \sum_i E_x^i \int \left(\varepsilon_{x0} \frac{R}{\rho} + R \frac{\Delta d\varphi \bar{z}}{\rho} \right) dA. \tag{25}$$

The sum is divided into two parts:

$$N = \sum_i E_x^i \left[\varepsilon_{x0} \int \frac{R}{\rho} dA + R \int \left(\frac{\Delta d\varphi \bar{z}}{\rho} \right) dA \right]. \tag{26}$$

The first part is:

$$Rb \sum_i E_x^i \left[In\rho \right]_{R_i}^{R_{i+1}} \varepsilon_{x0},$$

in the second part $\bar{z}$ will be replaced by $\boldsymbol{\rho}$-$\boldsymbol{R}$ and $\Delta d\boldsymbol{\varphi}$ and $\boldsymbol{R}$ taken before the summation:

$$R \Delta d\varphi \underbrace{\sum_i E_x^i \int \left(\frac{(\rho - R)}{\rho} \right) dA}_{0} = 0.$$

Now you can see that the expression is zero, because it was the condition which was used to determine R. This allows to calculate the axial force:

$$N = Rb \sum_i E_x^i \left[In\rho \right]_{R_i}^{R_{i+1}} \varepsilon_{x0}. \tag{27}$$

The term

$$AA = Rb\sum_i E_x^i\left[ln\rho\right]_{R_i}^{R_{i+1}}\varepsilon_{x0}, \qquad (28)$$

is called axial stiffness.

$$N = AA\varepsilon_{x0}. \qquad (29)$$

BENDING MOMENT M_Y

$$M_y = \int \sigma_x \bar{z} dA. \qquad (30)$$

Introducing equation (24) and Hooke's law into quation (30) leads to

$$M_y = \sum_i E_x^i \int \left(\varepsilon_{x0}\frac{R}{\rho}\bar{z} + \Delta d\varphi \bar{z}\frac{R}{\rho}\bar{z}\right)dA. \qquad (31)$$

Rearranging and substituting $\bar{z} = \boldsymbol{\rho}\text{-}\boldsymbol{R}$ leads to

$$M_y = b\sum_i E_x^i \int \left[\varepsilon_{x0}\frac{R}{\rho}(\rho - R) + \Delta d\varphi(\rho - R)^2\frac{R}{\rho}\right]d\rho$$

The integral has two parts, the first is

$$bR\varepsilon_{x0}\sum E_x^i \int \frac{\rho - R}{\rho}d\rho.$$

This part is zero because of equation (19). Therefore bending moment is:

$$M_y = \Delta d\varphi bR\sum E_x^i \int \frac{(\rho - R)^2}{\rho}d\rho.$$

and leads to

$$M_y = \underbrace{\Delta d\varphi bR\sum E_x^i\left[\frac{1}{2}\rho^2 - 2R\rho + R^2 \ln\rho\right]_{R_{i+1}}^{R_i}}_{DD} \qquad (32)$$

The bending stiness is:

$$DD = bR\sum E_x^i\left[\frac{1}{2}\rho^2 - 2R\rho + R^2 ln\rho\right]_{R_{i+1}}^{R_i}. \qquad (33)$$

With equation (32) and equation (33) the bending moment gives:

$$M_y = \Delta d\varphi\ DD. \qquad (34)$$

INTERNAL FORCES STRAIN RELATIONSHIP

Equations (30) describes the relations between the forces and strain, equation (34) of bending moment and curvature. There is no coupling between axial force and bending or between bending moment and strain. This is due the fact that the origin of coordinate system was not put in the geometric centroid of the cross section but in the neutral axis of a pure bending loaded beam.

Strain in neutral axis and $\Delta d\boldsymbol{\varphi}$ is computed by:

$$\varepsilon x0 = \frac{n}{AA}, \qquad (35)$$

and

$$\Delta d\varphi = \frac{M_y}{DD}. \qquad (36)$$

The normal stress $\boldsymbol{\sigma}_x$ is calculated by equation (24) in combination with Hooke's law:

$$\sigma_x = E_x^i\left(\varepsilon_{x0}\frac{R}{\rho} + \Delta d\varphi R\frac{\rho - R}{\rho}\right). \qquad (37)$$

Stresses in and perpendicular to ber and shear stress are calculated by equation (3). The stresses in material system are the basic to compute Tsai-Wu's failure criteria criterion [1] or Puck's failure criteria [3] for assessing the strength.

CONCLUSION

A loop consist of 8 layers of glas ber and epoxy resin with the symmetrical stacking: 0/90/45/-45/-45/45/90/0. Each layer of the beam has the thickness of $\boldsymbol{t}_i$ = 0:5 mm, the inner radius is $\boldsymbol{r}_i$ = 3 mm. The loading is axial force of $\boldsymbol{N}$ = 1000 N and bending moment $\boldsymbol{M}_y$ = 6 000 Nmm. The material is described with:

ber volume ratio $\boldsymbol{\varphi}_\upsilon$ = 0,6,

modulus of elasticity in ber direction:

$\boldsymbol{E}_1$ = 44 500 N/mm^2,

modulus of elasticity perpendicular to ber direction:

$\boldsymbol{E}_2$ = 12 500 N/mm^2,

shear modulus: $\boldsymbol{G}_{12}$ =6000 N/mm^2,

major Poisson's ratio: $\boldsymbol{\upsilon}_{12}$=0,28,

width $\boldsymbol{b}$ = 10 mm.

Evaluation of equation (21) gives $\boldsymbol{R}$ = 4,61 mm. The axial stiness according equation (28) gives $\boldsymbol{AA}$ = 8,8 10^5 N, the bending stiness $\boldsymbol{DD}$ = 1,6037 10^6 Nmm^2. In Fig. 2 the stresses $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\tau}_{12}$ in the material system over the radius are shown. In the 0-degree layers (rst and last layer) you can see the non-linear course of the stress ($\boldsymbol{\sigma}_1$) is evident. In the 90 layers (second and seventh layer) there are only stresses perpendicular to the ber direction ($\boldsymbol{\sigma}_2$) while the stress in ber direction and shear stress vanish.

CONCLUSION

In the present paper the generalization of the theory of isotropic curved beams has been extended to anisotropic materials. Of special signi cance is the choice of the origin of the coordinate system. If one put it in the neutral axis no coupling of normal force and bending moment occurs.

REFERENCES

[1] Halpin, Primer on Composit Materials Analysis, Lancaster, PA, 1984
[2] Leipholz, Festigkeitslehre für den Konstrukteur, Suttgart 1968
[3] Puck, Festigkeitsanalyse von Faser-Matrix-aminaten, München, Wien, 1996
[4] Michaeli/Huybrechts/Wegener, Dimensionieren mit Faserverbundwerkstoffen, München, Wien,1994
[5] Altenbach, Altenbach, Kissing, Mechanics of Composite Structural Elements, Springer, Berlin, 2004
[6] Michaeli/Wegener: Einführung in die Technologie der Faserverbundwerkstoe, München, Wien, 1989.
[7] Fleming, Faserverbundbaueise, Berlin, Springer, 1995
[8] Schürmann, Konstruieren mit Faser-Verbundwerkstoen, Berlin, Springer, 2005