A Construction of H -antimagic Graphs

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Abstract: Let $G = (V, E)$ be a finite simple graph with p vertices and q edges. An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i , $i=1, 2, \dots, t$. If every subgraph H_i is isomorphic to a given graph H , then the graph G admits an H -covering. Such a graph G is called (a, d) - H -antimagic if there is a bijection $f: V \cup E \rightarrow \{1, 2, \dots, p+q\}$ such that for all subgraphs H' of G isomorphic to H , the sum of the labels of all the edges and vertices belonging to H' constitutes an arithmetic progression with the initial term a and the common difference d . When $f(V) = \{1, 2, \dots, p\}$, then G is said to be *super (a, d) - H -antimagic*; and if $d = 0$ then G is called H -supermagic.

We will exhibit an operation on graphs which keeps super H -antimagic properties. We use a technique of partitioning sets of integers for the construction of the required labelings.

Keywords: H -covering, (a, d) - H -antimagic graph, super (a, d) - H -antimagic graph, partition of set.

1. Introduction

An *edge-covering* of a finite and simple graph G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i , $i=1, 2, \dots, t$. In this case we say that G admits an $(H_1, H_2, \dots, H_t)$ -*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H , then the graph G admits an H -covering. Suppose that a (p, q) -graph $G=(V, E)$ with p vertices and q edges admits an H -covering. The graph G is called (a, d) - H -antimagic if there exists a total labeling $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$ such that, for all subgraphs H' of G isomorphic to H , the H -weights,

$$wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e),$$

constitute an arithmetic progression $a, a+d, a+2d, \dots, a+(t-1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H . Moreover, G is said to be *super (a, d) - H -antimagic* if the smallest possible labels appear on the vertices. If G is a (super) (a, d) - H -antimagic graph then the corresponding total labeling f is called a (super) (a, d) - H -antimagic labeling. For $d = 0$, a (super) (a, d) - H -antimagic graph is called H -magic and H -supermagic, respectively.

The H -(super)magic graph was first introduced by Gutiérrez and Lladó in [9]. They proved that some classes of connected graphs are H -supermagic, for example, the stars $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$ -supermagic for some h . They also proved that the path P_n and the cycle C_n are P_h -supermagic for some h . Lladó and Moragas [15] investigated C_n -(super)magic graphs and proved that

wheels, windmills, books and prisms are C_h -magic for some h . Some results on C_n -supermagic labelings of several classes of graphs can be found in [19]. Maryati et al. [17] gave P_h -(super)magic labelings of some trees such as shrubs, subdivision of shrubs and banana tree graphs. Other examples of H -supermagic graphs with different choices of H have been given by Jeyanthi and Selvagopal in [12]. Maryati et al. [18] investigated the G -supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_h -supermagic for some c and h .

The (a,d) - H -antimagic labeling was introduced by Inayah et al. [10]. In [11] Inayah et al. investigated the super (a,d) - H -antimagic labelings for some shackles of a connected graph H .

For $H \cong K_2$, (super) (a,d) - H -antimagic labelings are also called simply (super) (a,d) -edge-antimagic total labelings. These labelings are the generalization of the edge-magic and super edge-magic labelings that were introduced by Kotzig and Rosa [13] and Enomoto et al. [7], respectively. For further information on (super) edge-magic labelings, see [4, 5, 8, 16].

The (super) (a,d) - H -antimagic labeling is related to a super d -antimagic labeling of type $(1,1,0)$ of a plane graph that is the generalization of a face-magic labeling introduced by Lih [14]. Further information on super d -antimagic labelings can be found in [1, 2, 3, 6].

In this paper we show one operation on graphs which keeps super H -antimagic properties. We use a technique of partitioning sets of integers for a construction of the required labelings.

2. Constructions Using Partitions of Integers

In this section we examine the existence of the super H -antimagic labelings for graphs obtained by one graph operation. The constructions of labelings will be made using partitions of the sets of integers.

Consider the partition $\mathcal{P}_2^n$ of the set of integers $\{1, 2, \dots, 2n\}$ into n , $n \geq 2$, couples such that the sums of the numbers in all couples are the same. If $\mathcal{P}_2^n(i)$ denotes the i th couple in the partition $\mathcal{P}_2^n$ then, for example,

$$\mathcal{P}_2^n(i) = \{a_i, b_i\} = \{i, 2n+1-i\},$$

where $i=1, 2, \dots, n$. For the sum of the two numbers

in the i th couple we have

$$\sum \mathcal{P}_2^n(i) = a_i + b_i = 2n+1$$

for $i=1, 2, \dots, n$.

A similar idea can be also used for a partition of the set of kn consecutive integers into k -tuples.

Let n , k and i be positive integers. We will consider the partition $\mathcal{P}_k^n$ of the set $\{1, 2, \dots, kn\}$ into n , $n \geq 2$, k -tuples such that the sum of the numbers in the i th k -tuple is always the same and equal to the constant $k(1+kn)/2$, where $i=1, 2, \dots, n$. Using the divisibility we have that if k is odd then n has to be odd too.

Let us consider the partition $\mathcal{P}_3^n$ of the set of integers $\{1, 2, \dots, 3n\}$, n odd, into triples such that the i th triple in the partition is defined in the following way

$$\begin{aligned} \mathcal{P}_3^n(i) &= \{a_i, b_i, c_i\} \\ &= \begin{cases} \left\{ \frac{n+i}{2}, n + \frac{i+1}{2}, 3n+1-i \right\} \\ \text{for } i \equiv 1 \pmod{2}, \\ \left\{ \frac{i}{2}, \frac{3n+1+i}{2}, 3n+1-i \right\} \\ \text{for } i \equiv 0 \pmod{2}. \end{cases} \end{aligned}$$

It is easy to see that the sum of all numbers in the i th triple is equal to

$$\sum \mathcal{P}_3^n(i) = a_i + b_i + c_i = \frac{3(3n+1)}{2} \quad (1)$$

for $i=1, 2, \dots, n$. Moreover, the minimum of the numbers in every triple is the number from the set $\{1, 2, \dots, n\}$.

Only for the purposes of this paper by the notation $\mathcal{P}_k^n(i) \oplus c$ we mean that the constant c is added to every element of $\mathcal{P}_k^n(i)$.

By $A-B$ we denote the difference of the set B from the set A .

Let G_2^k be a graph obtained from two isomorphic graphs G and G' by connecting corresponding vertices of G and G' with a matching, then subdivide every edge of the matching using k vertices. If a (p,q) -graph G has vertex set $V(G) = \{v_1, v_2, \dots, v_p\}$ then the graph G_2^k has the vertex set

$$V(G_2^k) = V(G) \cup V(G') \\ \cup \{v_i^j : i=1,2,\dots,p, j=1,2,\dots,k\}$$

and the edge set

$$E(G_2^k) = E(G) \cup E(G') \\ \cup \{v_i^1 v_i^j, v_i^j v_i^{j+1}, v_i^k v_i' : i=1,2,\dots,p, \\ j=1,2,\dots,k-1\}.$$

The graph G_2^k has $(k+2)p$ vertices and $2q + (k+1)p$ edges.

A useful property for finding H -antimagic labelings is given in the following lemma.

Lemma 1.

Let f be a super (a,d) - H -antimagic labeling of $G=(V,E)$ and let r, s be nonnegative integers. Then the labeling

$$g: V(G) \cup E(G) \rightarrow \{r+1, r+2, \dots, r+|V(G)|, \\ r+s+|V(G)|+1, \\ r+s+|V(G)|+2, \dots, \\ r+s+|V(G)|+|E(G)|\}$$

defined such that

$$g(v) = f(v) + r \quad \text{if } v \in V(G), \\ g(e) = f(e) + r + s \quad \text{if } e \in E(G)$$

has the property that

$$\{wt_g(H) : H \subset G\} = \{b, b+d, \dots, b+(t-1)d\},$$

where t is the number of all subgraphs in G isomorphic to H and b is a positive integer.

Proof.

Let

$$f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, |V(G)| + |E(G)|\}$$

be a super (a,d) - H -antimagic labeling of $G=(V,E)$ and let $H_1, H_2, \dots, H_t$ be all subgraphs of G isomorphic to H . Thus the set of all H -weights under the labeling f is

$$\{wt_f(H_i) : i=1,2,\dots,t\} \\ = \{a, a+d, \dots, a+(t-1)d\}. \quad (2)$$

For the H -weight of the subgraph H_i , $i=1,2,\dots,t$, under the labeling g we have

$$wt_g(H_i) = \sum_{v \in V(H_i)} g(v) + \sum_{e \in E(H_i)} g(e) \\ = \sum_{v \in V(H_i)} (f(v) + r) \\ + \sum_{e \in E(H_i)} (f(e) + r + s) \\ = \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e) \\ + r|V(H_i)| + (r+s)|E(H_i)| \\ = wt_f(H_i) + r|V(H_i)| \\ + (r+s)|E(H_i)|.$$

As all subgraphs H_i are isomorphic to H it holds

$$|V(H_i)| = |V(H)|, \\ |E(H_i)| = |E(H)|.$$

Thus $r|V(H_i)| + (r+s)|E(H_i)|$ is a constant for all $i=1,2,\dots,t$ and

$$wt_g(H_i) = \\ wt_f(H_i) + r|V(H)| + (r+s)|E(H)|. \quad (3)$$

According to the property (3) and by using (2) we get

$$\{wt_g(H_i) : i=1,2,\dots,t\} \\ = \{b, b+d, \dots, b+(t-1)d\},$$

where $b = a + r|V(H)| + (r+s)|E(H)|$.

□

Theorem 2.

Let G be a super (a,d) - H -antimagic graph of odd order containing t subgraphs isomorphic to H and

let k be a positive integer, $k \geq 1$. If the graph G_2^k contains exactly t subgraphs isomorphic to H_2^k then the graph G_2^k is super $(b, 2d)$ - H_2^k -antimagic.

Proof.

Let $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$ be a super (a, d) - H -antimagic labeling of a (p, q) -graph G of odd order and let $H_1, H_2, \dots, H_t$ be a family of all subgraphs of G isomorphic to H .

Clearly, the set of all H -weights is as follows.

$$\begin{aligned} \{wt_f(H_l) : l=1, 2, \dots, t\} \\ = \{a, a+d, \dots, a+(t-1)d\} \end{aligned} \quad (4)$$

and the smallest possible labels $1, 2, \dots, p$ appear on the vertices of G .

Let us consider the labeling g of the vertices and edges of G_2^k defined in the following way.

$$\begin{aligned} g(v) &= f(v) & \text{if } v \in V(G), \\ g(v') &= f(v) + p & \text{if } v' \in V(G'), \\ g(v_i^1) &= \min\{\mathcal{P}_3^p(i)\} \oplus 2p & \text{if } i=1, 2, \dots, p, \\ g(v_i^j) &= i + (j+1)p & \text{if } i=1, 2, \dots, p, \\ & & j=2, 3, \dots, k, \\ g(e) &= f(e) + (k+1)p & \text{if } e \in E(G), \\ g(e') &= f(e) + (k+1)p + q & \text{if } e' \in E(G'), \\ \{g(v_i v_i^1), g(v_i^k v_i')\} & & \\ &= \mathcal{P}_3^p(i) \oplus ((k+1)p + 2q) & \\ &\quad - (\min\{\mathcal{P}_3^p(i)\} \oplus ((k+1)p + 2q)) & \\ & & \text{if } i=1, 2, \dots, p, \\ g(v_i^{j-1} v_i^j) &= (2k+6)p + 2q + 1 - g(v_i^j) & \\ & & \text{if } i=1, 2, \dots, p, \\ & & j=2, 3, \dots, k. \end{aligned}$$

It is easy to see that g is a bijection from the vertex

set and edge set of G_2^k onto the set $\{1, 2, \dots, (2k+3)p + 2q\}$ and the vertices of G_2^k are labeled with the smallest possible labels.

For the H_2^k -weight of the subgraph $(H_l)_2^k$, $l=1, 2, \dots, t$, under the labeling g we get

$$\begin{aligned} wt_g((H_l)_2^k) &= \sum_{v \in V((H_l)_2^k)} g(v) + \sum_{e \in E((H_l)_2^k)} g(e) \\ &= \left(\sum_{v \in V(H_l)} g(v) + \sum_{v \in V(H_l')} g(v) \right. \\ &\quad \left. + \sum_{j=1}^k \sum_{i: v_i \in V(H_l)} g(v_i^j) \right) \\ &\quad + \left(\sum_{e \in E(H_l)} g(e) + \sum_{e \in E(H_l')} g(e) \right. \\ &\quad \left. + \sum_{i: v_i \in V(H_l)} (g(v_i v_i^1) + g(v_i^k v_i')) \right. \\ &\quad \left. + \sum_{j=2}^k \sum_{i: v_i \in V(H_l)} g(v_i^{j-1} v_i^j) \right) \\ &= wt_g(H_l) + wt_g(H_l') \\ &\quad + \sum_{i: v_i \in V(H_l)} (g(v_i^1) + g(v_i v_i^1) + g(v_i^k v_i')) \\ &\quad + \sum_{j=2}^k \sum_{i: v_i \in V(H_l)} (g(v_i^j) + g(v_i^{j-1} v_i^j)). \end{aligned}$$

According to (3) we get

$$\begin{aligned} wt_g(H_l) &= wt_f(H_l) + (k+1)p |E(H)|, \\ wt_g(H_l') &= wt_f(H_l) + p |V(H)| \\ &\quad + (kp + p + q) |E(H)|. \end{aligned}$$

According to (1) it holds

$$\begin{aligned} &\sum_{i: v_i \in V(H_l)} (g(v_i^1) + g(v_i v_i^1) + g(v_i^k v_i')) \\ &= \sum_{i: v_i \in V(H_l)} ((\min\{\mathcal{P}_3^p(i)\} \oplus 2p) \\ &\quad + \sum (\mathcal{P}_3^p(i) \oplus ((k+1)p + 2q)) \\ &\quad - (\min\{\mathcal{P}_3^p(i)\} \oplus ((k+1)p + 2q))) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i: v_i \in V(H_l)} \sum (\mathcal{P}_3^p(i)) + 2((k+2)p + 2q) |V(H_l)| \\
&= \frac{3(3p+1) |V(H_l)|}{2} + 2((k+2)p + 2q) |V(H_l)| \\
&= \frac{(4k+17)p + 8q + 3}{2} |V(H_l)| \\
&= \frac{(4k+17)p + 8q + 3}{2} |V(H)|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sum_{j=2}^k \sum_{i: v_i \in V(H_l)} (g(v_i^j) + g(v_i^{j-1} v_i^j)) \\
&= \sum_{j=2}^k \sum_{i: v_i \in V(H_l)} (g(v_i^j) + (2k+6)p + 2q + 1 - g(v_i^j)) \\
&= \sum_{j=2}^k \sum_{i: v_i \in V(H_l)} ((2k+6)p + 2q + 1) \\
&= ((2k+6)p + 2q + 1)(k-1) |V(H_l)| \\
&= ((2k+6)p + 2q + 1)(k-1) |V(H)|.
\end{aligned}$$

Thus, for $l=1,2,\dots,t$ we get

$$wt_g((H_l)_2^k) = 2wt_f(H_l) + A,$$

where

$$\begin{aligned}
A &= ((2k+2)p + q) |E(H)| \\
&+ \frac{(4k^2 + 12k + 7)p + (4k+4)q + 2k+1}{2} |V(H)|.
\end{aligned}$$

Using (4) we have

$$\begin{aligned}
&\{wt_g(H_l) : l=1,2,\dots,t\} \\
&= \{2a + A, 2a + A + 2d, \dots, 2a + A + 2(t-1)d\}.
\end{aligned}$$

This concludes the proof that the graph G_2^k is super $(2a+A, 2d)$ - H_2^k -antimagic. $\square$

3. Conclusion

In this paper we examined the existence of super H_2^k -antimagic labelings for graphs G_2^k obtained

from two isomorphic graphs G and G' by joining every couple of corresponding vertices $v \in V(G)$ and $v' \in V(G')$ by a path of length $k+1$.

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5. References and Notes

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